Econ 201C: Problem Set 1
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1. **Contracting with Externalities**

   Two firms ("agents") would like to invest in a city (e.g. real estate investments). The Govenment ("principal") is willing to let the agents invest in exchange for payments. Formally, the principal offers each agent $i \in \{1, 2\}$ a contract $\langle x_i, t_i \rangle$ which describes the investment level $x_i \geq 0$ and payment $t_i \geq 0$. The contracts are observed by both agents before they choose to accept.

   The principal obtains revenue $\pi = t_1 + t_2$. If agent i accepts the contract, he makes utility $u_i = v_i - t_i$, where
$$v_i = x_i - \frac{1}{2}x_i^2 + \alpha x_j.$$
   If agent i rejects the contract, $\langle x_i, t_i \rangle = \langle 0, 0 \rangle$ and so he makes utility $u_i = v_i = \alpha x_j$. Note that $\alpha > 0$ means the agents have positive externalities on each other, while $\alpha < 0$ means the agents have negative externalities on each other. We assume that $\alpha > -1$.

   (a) What is the Pareto efficient investment level?

   Aggregating the principal and the two agents utilities gives us a welfare function only dependent on the levels of investment x_1, x_2:
$$SWF(x_1, x_2) := \pi + u_1 + u_2 = (1+\alpha)x_1 - \frac{1}{2}x_1^2 + (1+\alpha)x_2 - \frac{1}{2}x_2^2.$$
   Using the FOC, we derive the efficient investment $x_1^E = x_2^E = 1 + \alpha$.

   (b) Suppose the principal offers a "multilateral" contract $\langle x_i, t_i \rangle$ to each agent simultaneously. Formally: (i) the principal offers each agent a contract $\langle x_i, t_i \rangle$. (ii) Both choose to accept or reject. (iii) If either agent rejects then both contracts are cancelled. What are the principal's optimal choices of x_1 and x_2?[1]

   The outside option for both agents is 0, since canceling their contract means the other cannot invest. The principal thus maximizes π subject to each agent's (binding) IR constraint:
$$\max_{x,t} t_1 + t_2 \quad \text{s.t.} \quad x_i - \frac{1}{2}x_i^2 + \alpha x_{-i} = t_i \quad \forall i \in \{1, 2\}.$$
   Substituting the IR into the objective yields $SWF(x_1, x_2)$, which is maximized at $x_1^\star = x_2^\star = 1 + \alpha$, and first-best is achieved.

   (c) Suppose the principal offers "bilateral" contract $\langle x_i, t_i \rangle$ to each agent simultaneously. Formally: (i) the principal offers each agent a contract $\langle x_i, t_i \rangle$. (ii) Both choose to accept or reject. (iii) If an agent rejects then the other agent's contract is unaffected. What are the principal's optimal choices of x_1 and x_2?

[1] We wish to characterize the principal's best equilibrium and are thus doing "partial implementation". There is trivially an equilibrium where both agents reject; we will ignore this.

Agent i's IR constraint is now

$$x_i - \frac{1}{2}x_i^2 - t_i \geq 0,$$

since the externality from the investment of agent j is always received, whether i accepts or rejects. Making these bind and substituting this again into the objective function of the principal yields

$$\max_x x_1 - \frac{1}{2}x_1^2 + x_2 - \frac{1}{2}x_2^2,$$

which is maximized at $x_1^\star = x_2^\star = 1$.

(d) How does the level of investment under multilateral and bilateral contracts depend on α? Provide an intuition.

The level of investment in the multilateral contract depends on α additively. Intuitively, say $\alpha > 0$, so the agents exert positive externalities on each other. Then, by enforcing a higher level of investment from both agents, they jointly receive more utility and are further incentivized to both accept the contract, even at higher levels of payment, since they receive nothing if they reject. Thus, the principal is able to gain more revenue (and vice versa when $\alpha < 0$). The level of investment in the bilateral contract is fixed at 1 and not dependent on α. Since each agent receives the externality from the other agent no matter their choice of accepting or rejecting, they only maximize over their own investment (from the perspective of the dual where agents have bargaining power), which yields investment of 1 each.

(e) How does the principal's profit under multilateral and bilateral contracts depend on α? Provide an intuition.

The principal's profit is $\pi_m = (1+\alpha)^2$ in the multilateral and $\pi_b = 1$ in the bilateral. If the agents have a positive externality on each other, the principal makes higher profits in the multilateral contract since they can enforce higher investment and charge higher payments, since each agent wants the other to stay. If the agents have negative externalities on each other, the principal makes lower profits in the multilateral contract since agents might not want the other to invest anymore and they are more incentivized to reject, so the principal has to charge lower payments. In the bilateral contract, agents are not considering the externality from the other, so the principal doesn't have to either when choosing revenue-maximizing investment levels.

2. **Observable but non-Verifiable Effort**

An agent chooses effort $a \in A$ with increasing, convex cost $c(a)$. Suppose output $q \sim f(q \mid a)$ increases in effort in the sense of first-order stochastic dominance. The agent is risk averse with utility $u(w(q)) - c(a)$, where $w(q)$ is their final compensation. The principal is risk-neutral with profits $q - w$. The agent has outside option $\bar{u}$, while the the firm has all the bargaining power.

(a) Suppose the principal can contract on the agent's effort. Characterize the first-best effort a^*, expected output q^*, and expected profit, Π^*.

The principal solves

$$\max_{a,w(q)} \mathbb{E}[q - w(q) \mid a] \quad \text{s.t.} \quad \mathbb{E}[u(w(q)) - c(a) \mid a] \geq \bar{u} \quad (IR).$$

Letting the IR bind, we have the Lagragian

$$\int_q (q - w(q) + \lambda u(w(q)))\, dF(q|a) - \lambda c(a) - \lambda \bar{u}$$

which we maximize state-wise with respect to w:

$$\max_w -w + \lambda u(w).$$

Taking the FOC thus yields the optimal wage schedule $w(q) = w^\star$ where $w^\star$ solves $1/u'(w^\star) = \lambda$. Moving to a, the binding IR can be written as

$$\begin{aligned} &u(w^\star) - c(a) = \bar{u}, \\ &w^\star = u^{-1}(c(a) + \bar{u}). \end{aligned}$$

Plugging this into the problem of the principal yields

$$\max_a \mathbb{E}[q \mid a] - u^{-1}(c(a) + \bar{u}).$$

Taking the FOC characterizes $a^\star$:

$$\int q \cdot f_a(q|a^\star)dq = \frac{c'(a^\star)}{u'(u^{-1}(c(a^\star) + \bar{u}))}.$$

Finally, the expected profit:

$$\Pi^\star = \mathbb{E}[q \mid a^\star] - u^{-1}(c(a^\star) + \bar{u}).$$

Now, suppose that both principal and agent can observe effort a, but cannot contract on it. We wish to show that the following option contract implements the first-best. The agent first buys the firm for Π^*. After the effort is chosen, but before output is realized, the principal then has the option to buy back the firm and obtain all the output for a price q^*.

(b) How does the principal's decision to exercise the option depend on the agent's choice of effort?

Since the principal is risk-neutral, the principal exercises when the agent's effort a satisfies

$$\mathbb{E}[q|a] \geq q^\star.$$

If we choose $q^\star = \mathbb{E}[q|a^\star]$, then since $F(\cdot|a)$ increases in effort, we have the cutoff rule that the principal exercises when

$$a \geq a^\star.$$

(c) Show that the agent will choose a^* and receive utility $\bar{u}$.

First consider the case where the agent chooses $a \geq a^\star$, so the principal exercises their option. Then, the total payoff for the agent is

$$\begin{aligned} u(q^\star - \Pi^\star) - c(a) &= u(q^\star - \mathbb{E}[q \mid a^\star] + u^{-1}(c(a^\star) + \bar{u})) - c(a) \\ &= u(u^{-1}(c(a^\star) + \bar{u})) - c(a) \\ &= \bar{u} + c(a^\star) - c(a). \end{aligned}$$

Because $c(\cdot)$ is increasing in a, the agent will want to choose the smallest subject to $a \geq a^\star$, i.e., $a^\star$, and thus receives $\bar{u}$. Now, consider the alternative if the agent shirks, i.e. chooses $a < a^\star$. The principal wouldn't exercise, so the agent would keeps the output. Thus, the agent has expected utility:

$$\mathbb{E}[u(q - \Pi^\star)|a] - c(a) \leq u(\mathbb{E}[q|a] - \Pi^\star) - c(a), \tag{1}$$

where the inequality follows by risk-aversion. By the definition of $\Pi^\star$, we also have for any $a \neq a^\star$

$$\mathbb{E}[q \mid a] - u^{-1}(c(a) + \bar{u}) \leq \Pi^\star,$$

and rearranging and applying the monotone u, we get that

$$u(\mathbb{E}[q|a] - \Pi^\star) \leq c(a) + \bar{u}.$$

Plugging this inequality into 1 gives

$$\mathbb{E}[u(q - \Pi^\star)|a] - c(a) \leq \bar{u},$$

meaning the agent's expected utility is bounded above by $\bar{u}$, which they receive guaranteed if choosing $a^\star$. Thus, the agent will not deviate. □

3. **Normal–Linear Model**

The following normal–linear model is regularly used in applied models. Given action $a \in \mathbb{R}$, output is $q = a + x$, where $x \sim N(0, \sigma^2)$. The cost of effort is $c(a)$ is increasing and convex. The agent's utility equals $u(w(q) - c(a))$,[2] while the principal's is $q - w(q)$. Suppose the agent's outside option is $u(0)$.

We make two large assumptions. First, the principal uses a linear contract:

$$w(q) = \alpha + \beta q$$

Second, the agent's utility is CARA, i.e.,

$$u(w) = -e^{-w}.$$

[2]Note that the cost of effort is inside the agent's utility. This is important for the "certainty equivalent approach" we use below.

(a) Suppose $w \sim N(\mu, \sigma^2)$. Denote the certainty equivalent of w by $\bar{w}$, where

$$u(\bar{w}) = E[u(w)].$$

Show that $\bar{w} = \mu - \sigma^2/2$.

Since w is a normal RV and u is CARA, we have that

$$\begin{aligned} E[u(w)] &= \int_{\mathbb{R}} -\exp(-x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\ &= -\int \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2+2\sigma^2 x}{2\sigma^2}\right)dx. \end{aligned}$$

We can complete the square as such:

$$\begin{aligned} (x-\mu)^2 + 2\sigma^2 x = x^2 - 2\mu x + \mu^2 + 2\sigma^2 x &= x^2 + 2(\sigma^2-\mu)x + \mu^2 \\ &= (x+\sigma^2-\mu)^2 + \mu^2 - (\sigma^2-\mu)^2. \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}[u(w)] &= -\int \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x+\sigma^2-\mu)^2+\mu^2-(\sigma^2-\mu)^2}{2\sigma^2}\right)dx \\ &= -\exp\left(-\frac{\mu^2-(\sigma^2-\mu)^2}{2\sigma^2}\right)\underbrace{\int \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x+\sigma^2-\mu)^2}{2\sigma^2}\right)dx}_{=1 \text{ since density}} \\ &= -\exp\left(-\frac{\mu^2-(\sigma^2-\mu)^2}{2\sigma^2}\right) \\ &= -\exp\left(-\frac{2\mu\sigma^2-\sigma^4}{2\sigma^2}\right) \\ &= -\exp\left(-\mu+\frac{\sigma^2}{2}\right) \\ &= u(\mu-\frac{\sigma^2}{2}), \end{aligned}$$

so the certainty equivalent is $\bar{w} = \mu - \frac{\sigma^2}{2}$.

(b) Suppose effort is unobservable. The principal's problem is

$$\max_{w(q),a} E[q - w(q)]$$

s.t.

$$\begin{aligned} E[u(w(q)-c(a)) \mid a] &\geq u(0) \\ a \in \arg\max_{a^0\in\mathbb{R}} E[u(w(q)&-c(a^0)) \mid a^0]. \end{aligned}$$

Use the first order approach to (implicitly) characterize the optimal contract (α, β, a).[3]

[3]Hint: write utilities in terms of their certainty equivalent.

First note that $q|a \sim \mathcal{N}(a, \sigma^2)$. Thus, we can rewrite the agent's utility in terms of the certainty equivalent:

$$\begin{aligned} E[u(w(q) - c(a)) \mid a] &= E[u(\underbrace{\alpha + \beta q - c(a) \mid a}_{\sim \mathcal{N}(\alpha+\beta a - c(a), \beta^2\sigma^2)})] \\ &= u\left(\alpha + \beta a - c(a) - \frac{\beta^2\sigma^2}{2}\right). \end{aligned}$$

Since u is monotone, a is chosen to maximize the argument (IC), and by the FOC, we get the ICFOC:

$$\beta = c'(a).$$

The principal chooses the wage α, β and a such that (IR) binds, i.e.

$$\alpha + \beta a - c(a) - \frac{\beta^2\sigma^2}{2} = 0.$$

Now, writing the principals objective in terms of the linear contract and substituting in IR and the ICFOC, we have

$$\begin{aligned} E[q - w(q)|a] &= E[q|a] - E[\alpha + \beta q|a] \\ &= (1-\beta)a - \alpha \\ &= (1-\beta)a - c(a) - \frac{\beta^2\sigma^2}{2} + \beta a \\ &= a - c(a) - \frac{(c'(a))^2\sigma^2}{2}. \end{aligned}$$

Taking the FOC w.r.t. a, we have that the optimal action $a^\star$ is characterized by

$$1 - c'(a^\star) - c'(a^\star)c''(a^\star)\sigma^2 = 0,$$

and also, $\beta^\star = c'(a^\star)$ and $\alpha^\star = c(a^\star) + \frac{(\beta^\star)^2\sigma^2}{2} - \beta^\star a^\star$.

4. **Insurance Contracts**

A risk-averse agent starts with wealth W_0 and may have an accident costing x of their wealth, where x is contractible. The agent has access to a perfectly competitive market of risk-neutral insurers who offer payments $R(x)$ net of any insurance premium.[4]

The agent can affect the outcome by choosing action $a \in \mathbb{R}_+$. The distribution of x is as follows

$$f(0 \mid a) = 1 - p(a) \tag{1}$$

$$f(x \mid a) = p(a)g(x) \qquad \text{for } x > 0 \tag{2}$$

where $\int g(x)\,dx = 1$. The agent has increasing, concave, utility $u(\cdot)$. The cost is given by increasing, convex, differentiable function, $c(a)$. The function $p(a)$ is decreasing, convex and differentiable. The agent's utility is thus given by $u(W_0 - x + R(x)) - c(a)$, while the firm's profit is $-R(x)$.

[4]Since the market is competitive, the agent has the bargaining power and proposes the contract to maximize her utility.

(a) Suppose there is no insurance market. What action $\hat{a}$ does the agent take?

The agent solves

$$\max_a U_a = E[u(W_0 - x)|a] - c(a) = (1 - p(a))u(W_0) + p(a)\int_0^\infty u(W_0 - x)g(x)dx - c(a).$$

Taking the FOC w.r.t. a, we get that $\widehat{a}$ is characterized by

$$\left(-u(W_0) + \int_0^\infty u(W_0 - x)g(x)dx\right)p'(\widehat{a}) = c'(\widehat{a}).$$

(b) Suppose a is contractible. Characterize the first–best payment schedule $R(x)$ and the effort choice, a^*.

The agent chooses insurance contract $(a, R(x))$ to solve

$$\max_{a,R(x)} (1 - p(a))u(W_0 + R(0)) + p(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - c(a)$$

subject to the insurers' IR constraint(s) (they want to make at least zero expected profit, since risk-neutral):

$$(1 - p(a))R(0) + p(a)\int_0^\infty R(x)g(x)dx \geq 0.$$

Since the agent is risk-averse and the insurers are risk-neutral, we achieve first best when all risk is transferred to insurers, i.e., the agent is completely insured and has the same level of wealth, call it $\bar{w}$, for any accident x. We know for $x = 0$ that $\bar{w} = W_0 + R(0)$, so

$$\bar{w} = W_0 + R(0) = W_0 - x + R(x),$$

giving the optimal insurance schedule $R(x) = R(0) + x$. Plugging this into the IR constraint gives

$$(1 - p(a))R(0) + p(a)\int_0^\infty (R(0) + x)g(x)dx = 0$$

and solving for $R(0)$ gives

$$R(0) = -p(a) \cdot \int_0^\infty x \cdot g(x)dx,$$

so $R^\star(x) = x - p(a) \cdot \int_0^\infty x \cdot g(x)dx$. Under this full insurance, the agent simply maximizes

$$\max_a u\left(W_0 - p(a) \cdot \int_0^\infty x \cdot g(x)dx\right) - c(a),$$

where the optimal effort $a^\star$ is characterized by the FOC:

$$c'(a^\star) = u'\left(W_0 - p(a) \cdot \int_0^\infty x \cdot g(x)dx\right)\left(-p'(a^\star) \cdot \int_0^\infty x \cdot g(x)dx\right).$$

(c) Suppose a is not observable. Suppose the agent chooses a contract to maximize his utility subject to the competitive market of insurers breaking even.[5] Characterize the second–best payment schedule $R(x)$.

The agent chooses insurance contract anticipating their unobserved effort choice, i.e. anticipating solving

$$\max_{a} (1-p(a))u(W_0+R(0))+p(a)\int_0^\infty u(W_0-x+R(x))g(x)dx - c(a).$$

The FOC gives the optimal effort condition:

$$-p'(a)\left(u(W_0+R(0))-\int_0^\infty u(W_0-x+R(x))g(x)dx\right)=c'(a)$$

Note the agent's action only affects the probability that an accident occurs ($x>0$), and not the actual severity of the accident, so the second-best insurance schedule for a risk-averse agent and risk-neutral insurers should insure a constant level of wealth for the agent if an accident occurs ($\bar{w}_1$), and another constant level of wealth for the agent if an accident doesn't occur ($\bar{w}_0$). Thus, for $x>0$:

$$W_0-x+R(x)=\bar{w}_1$$

and $R(x)=x+\bar{w}_1-W_0 =: x+k$ for $x>0$, while for $x=0$, $R(0)=\bar{w}_0-W_0 =: r$. If $\bar{w}_1=\bar{w}_0$, then by the optimal effort condition, we would have $c'(a)=0$, but c is increasing. Thus, since $p'<0$, we must have $\bar{w}_1<\bar{w}_0$. Plugging this into the break-even constraint for the insurers, we have

$$(1-p(a))r+p(a)\int_0^\infty (x+k)g(x)dx=0,$$

and plugging this into the ICFOC, we have

$$-p'(a)\left(u(W_0+r)-u(W_0+k)\right)=c'(a).$$

Subject to these constraints, the agent then solves

$$\max_{a,<r,k>} (1-p(a))u(W_0+r)+p(a)u(W_0+k)-c(a).$$

where $\langle r^\star,k^\star\rangle$ characterizes the second-best insurance contract.

(d) Interpret the second–best payment schedule. Would the agent ever have an incentive to hide an accident? (i.e. report $x=0$ when $x>0$).

The second-best payment schedule insures the agent on the severity of an accident, conditional on it occurring, but not whether an accident actually happens. This is shown through the guaranteed wealth for when an accident occurs and when it doesn't. The difference in wealth levels is to incentivize the agent to exert more effort to reduce

[5] Note that this is the reverse of the usual principal-agent where the principal offers the contract. These problems are duals of each other, so the difference is superficial.

accident probability to secure the higher wealth $\bar{w}_0 = W_0 + r$ and not incur the lower wealth $\bar{w}_1 = W_0 + k$. This difference in payment schedules between whether an accident occurs or not may also incentivize an agent to hide a small accident. Suppose $x > 0$. Then reporting would give wealth $\bar{w}_1 = W_0 + k$, while not reporting would give wealth $\bar{w}_0 - x = W_0 + r - x$. Thus, if $x < r - k$, it would be better for the agent to not report.